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THE CALCULUS OF VARIATIONS : DERIVATION OF SOME OF THE FUNDAMENTAL WEIERSTRASSIAN FORMULÆ.

By Dr. Harris Hancock, Chicago, Ill.

1. Consider the integral

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt,$$

where F is a given function of the four arguments x, y, x', y', the quantities x' and y' being written for $\frac{dx}{dt}$ and $\frac{dy}{dt}$; farther we must regard F as a *one valued* regular function of these four arguments, one valued not in the analytical sense, but only for real values of the arguments. x and y are defined for the whole plane or for a connected portion of it, while x' and y' are to be considered as variables that are not limited, since they determine the direction of the tangent, and it is supposed that we may go in any direction from the point x, y. In our problem new assumptions are made regarding x and y, but not regarding x' and y'. We further assume that the functions x, y, x' and y' are capable of being differentiated, and that the curve is regular throughout its whole extent, or is composed of regular portions. Consequently x and y considered as functions of t and written $x(t)$, $y(t)$, are one valued regular functions of t throughout its whole extent or throughout the regular portions ; in the latter case we shall limit ourselves to one regular portion. If we did not make this assumption, the curve could not be the object of mathematical investigation, since there is no method of treating in their generality irregular curves ; and, if we wish the rules of the differential and integral calculus to be sufficient, then we must first apply our investigation to such functions, to which the rules are applicable without any limitation ; that is, to functions having the above properties.

2. If we find a curve which is regular and which satisfies the conditions of the problem, then it remains yet as a supplement to prove that it is the only curve which satisfies the condition of the problem.*

For example, it is found that of all regular closed curves of given perimeter, the circle is the one which incloses the greatest surface ; *à priori*, however, it is not known that a regular curve satisfies the problem. We know that of all polygons with a given number of sides and having a given perimeter the regular polygon has the greatest surface, and we thus come to the conclusion that the circle, to which the polygon approaches when the number of sides is increased, will have the greatest surface of all the closed curves; however, no one will recognize in this a rigorous proof, and in fact there yet remains a peculiar artifice to prove this property of the circle. We may at the same time point out the difficulty from which the calculus of variations at the present time is not entirely free.

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* Extracts from Weierstrass' lectures on "The Calculus of Variations."

3. The chief difficulty in all analysis consists in giving a strenuous proof that the necessary conditions that have been found for the existence of a certain property are also sufficient. In analytical researches we make conclusions in the following manner: If the analytical quantities exist which are required through the problems that have been set, then they must have certain properties; this gives the necessary conditions for the sought functions. It remains yet reciprocally to prove: if the conditions for an analytical object (curve, surface, etc.) are fulfilled, then the analytical object satisfies the conditions of the problem.

We therefore presuppose in our investigations, that the required functions are regular in their whole extent, and we seek the necessary conditions for the function which are given from the problems. Finally we will free ourselves from the limitations as far as it is possible, and see whether also the functions which have been found correspond to the conditions of the problem.

4. We will now prove an important property of the function F (Art. 1). In all the problems* which have been discussed the following is to be observed: the value of the integral, which is to be a minimum, depends in all cases only upon the form of the curve which is to be determined, not upon the manner in which x, y are represented as functions of a quantity t.

For example, if in the first problem we write the integral in the form

$$\int_{x_0}^{x_1} y\sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx,$$

then t is exactly equal to x, and it is clear that the value of this integral is the same as it was for the previous form (p. 183).

If we write for t any function of another quantity τ of such a nature that to the values t_0 and t_1 of t the values τ_0 and τ_1 of τ correspond, and that the curve with increasing τ will be traversed in the same direction as in the first case with increasing t, then the integral must remain unaltered, if it is to be independent of the manner in which x, y are represented as functions of a quantity t; that is, we must have :

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* See Annals of Mathematics, Vol. IX, p. 180, et seq.

1)
$$\int_{t_0}^{t_1} F\left[x, y, \frac{dx}{dt}, \frac{dy}{dt}\right] dt = \int_{\tau_0}^{\tau_1} F\left[x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right] d\tau \,.$$

The simplest function of this kind that we can write for t, is $t = k\tau$, where k represents any arbitrary but *positive* quantity. Hence considering x, y as functions of τ in the left hand side of 1) we have

$$\int_{t_0}^{t_1} F\left[x, y, \frac{dx}{dt}, \frac{dy}{dt}\right] dt = \int_{\tau_0}^{\tau_1} F\left[x, y, \frac{1}{k}\frac{dx}{d\tau}, \frac{1}{k}\frac{dy}{d\tau}\right] k d\tau \,;$$

hence

2)
$$\int_{\tau_0}^{\tau_1} F\left[x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right] d\tau = \int_{\tau_0}^{\tau_1} F\left[x, y, \frac{1}{k}\frac{dx}{d\tau}, \frac{1}{k}\frac{dy}{d\tau}\right] k d\tau \,.$$

Since this equation must be true for any arbitrary positive value of k, which however is not necessarily a constant, but may be any continuous positive function, it follows that the functions to be integrated must themselves be equal for every positive value of k ; and consequently

$$F\left[x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right] = kF\left[x, y, \frac{1}{k}\frac{dx}{d\tau}, \frac{1}{k}\frac{dy}{d\tau}\right] ;$$

or

3)
$$F(x, y, kx', ky') = kF(x, y, k', y') \,.$$

That is, if the integral I is to depend only upon the form of the curve (or, in other words, upon the analytical connection between x and y), then $F(x, y, x', y')$, with regard to x' and y', must be a homogeneous function of the first degree. This condition is also sufficient ; for consider x, y first expressed as function of a quantity t and then as functions of a quantity τ, and if these functions are of such a nature that the curve is traversed from the beginning point to the end point when t takes all values from t_0 and t_1, and τ all values from τ_0 to τ_1, then we can write $\frac{dt}{d\tau} = k$, if t increases at the same time as τ. Since k is a positive quantity, the correctness of the expression 2) follows from the existence of 3) and at the same time also the correctness of 1).

5. The condition that $F(x, y, x', y')$ must be a homogeneous function of the first degree with regard to x' and y' is generally expressed in another manner.

In fact it is nothing else than the condition of integrability of F. For if $F(x, y, x', y')\,dt$ is to be an exact differential, so that say

$$F(x, y, x', y') = d\Phi\,,$$

then the equation

$$F(x, y, x', y') = \frac{\partial\varphi}{\partial x}x' + \frac{\partial\varphi}{\partial y}y' + \frac{\partial\varphi}{\partial x'}x'' + \frac{\partial\varphi}{\partial y'}y''$$

must exist identically.

Since no second differential quotient is present in F, it follows that

$$\frac{\partial\varphi}{\partial x'} = 0 \text{ and } \frac{\partial\varphi}{\partial y'} = 0\,,$$

and therefore

$$F(x, y, x', y') = \frac{\partial\varphi}{\partial x}x' + \frac{\partial\varphi}{\partial y}y'\,.$$

But this is nothing more than that F is a homogeneous function of the first degree in x', y'.

This is everywhere the case in the examples given above (pp. 183 et seq.).

6. If the curve is of such a nature that one may regard the one coordinate as a one valued function of the other, and in such a way that for every value of x between two limits x_0 and x_1, there corresponds only one definite value of y, and that x continuously increases when we traverse the curve from the beginning point to the end point, then we may choose for t the quantity x itself, and therefore with the integral in the form :

4)
$$I = \int_{x_0}^{x_1} F\left[x, y, 1, \frac{dy}{dx}\right] dx\,,$$

as it is usually written.

7. This representation is not always true, since the above conditions which are necessary are not always fulfilled ; for example, in the fourth problem (p. 186) we must distribute the yet unknown curve into several parts, which does not appear convenient.

On the other hand, a representation such as given above is always possible, if we introduce the quantity t ; since one could introduce as the variable t the arc s of the curve measured from the beginning point. Besides in the form 4) it sometimes unavoidably happens that $\frac{dy}{dx}$ and consequently F becomes infinite within the limits of integration ; on the other hand it is generally possible so to choose t that this is not the case.

For these reasons we* prefer, in spite of the fact that many developments become more cumbrous, to represent the problem of the calculus of variations in the form:

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* With Prof. Weierstrass.

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt\,;$$

for on the other hand its great symmetry overbalances the fault given above.

8. *Analytical condition for $F(x, y, x', y')$.*

In the relation (Art. 4)

$$F(x, y\,;\ kx', ky') = kF(x, y, x', y')\,,$$

write $k = 1 + h$, then is

$$F(x, y\,;\ (1 + h)\,x', (1 + h)\,y') = (1 + h)\,F(x, y, x', y')$$

or:

$$F + \left[\frac{\partial F}{\partial x'}\,x' + \frac{\partial F}{\partial y}\,y'\right] h + h^2(\ldots) = F + hF$$

where

$$F = F(x, y, x', y')\,.$$

Therefore equating the coefficients of h:

1)
$$F = x'\frac{\partial F}{\partial x'} + y'\frac{\partial F}{\partial y'}\,, \ldots, \qquad (1)$$

which is another condition of homogeneity which F must have.

9. Differentiate 1) first with regard to x' and then with regard to y', which is allowable, since F is a regular function, and x', y' vary in a continuous manner, and we have

$$\left.\begin{aligned} &\alpha) \quad \frac{\partial^2 F}{\partial x'^2}\,x' + \frac{\partial^2 F}{\partial y'\partial x'}\,y' = 0 \\ &\beta) \quad \frac{\partial^2 F}{\partial x'\partial y'}\,x' + \frac{\partial^2 F}{\partial y'^2}\,y' = 0 \end{aligned}\right\}\ldots\ldots \qquad (2)$$

Hence from α)

$$\frac{\partial^2 F}{\partial x'^2} : \frac{\partial^2 F}{\partial y'\partial x'} = y' : -x' = y'^2 : -x'y'$$

and from β)

$$\frac{\partial^2 F}{\partial x'\partial y'} : \frac{\partial^2 F}{\partial y'^2} = -y' : x' = -x'y' : x'^2\,.$$

Therefore

$$\frac{\partial^2 F}{\partial x'^2} : \frac{\partial^2 F}{\partial y'\partial x'} : \frac{\partial^2 F}{\partial y'^2} = y'^2 : -x'y' : x'^2\,;$$

and if F_1 denotes a factor of proportionality we have:

3)
$$\frac{\partial^2 F}{\partial x'^2} = F_1 y'^2\,; \quad \frac{\partial^2 F}{\partial x'\partial y'} = -F_1 x'y'\,; \quad \frac{\partial^2 F}{\partial y'^2} = F_1 y'^2\,.$$

And consequently

$$\frac{\partial^2 F}{\partial x'^2} \Big/ y'^2 = \frac{\partial^2 F}{\partial x' \partial y'} \Big/ (-x'y') = \frac{\partial^2 F}{\partial y'^2} \Big/ y'^2 = F_1 .$$

F is of the 1st dimension in x' and y'; $\frac{\partial F}{\partial x'}$, $\frac{\partial F}{\partial y'}$ are of the dimension 0 in x' and y' and $\frac{\partial^2 F}{\partial y'^2}$, $\frac{\partial^2 F}{\partial x' \partial y'}$, $\frac{\partial^2 F}{\partial y'^2}$ are of the $-$ 1st dimension; consequently F_1 is of the dimension $-$ 3 in x' and y'.

10. *A Closer Study of the First Variation.*

We return to the integral (see Art. 1):

$$1) \qquad I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt ,$$

and make the supposition, that along the curve P_0P_1, the quantities x, y, and their first derivatives with regard to t are continuous functions of t; and we now no longer* assume that the points P_0 and P_1 remain fixed.

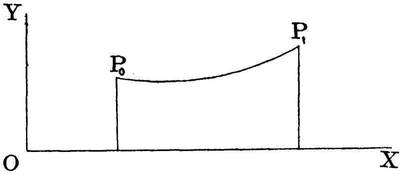

FIG. 1.

We make the substitutions:

$$\begin{array}{c||l} x & x + \varepsilon\xi + \frac{\varepsilon^2}{1\,.\,2}\xi^2 + \ldots , \\ y & y + \varepsilon\eta + \frac{\varepsilon^2}{1\,.\,2}\eta^2 + \ldots , \\ x' & x' + \varepsilon\xi' + \frac{\varepsilon^2}{1\,.\,2}\xi'^2 + \ldots , \\ y' & y' + \varepsilon\eta' + \frac{\varepsilon^2}{1\,.\,2}\eta'^2 + \ldots , \end{array}$$

where ξ, η, ξ', η' are continuous functions of t; and ξ, η are not zero at the points t_0 and t_1.

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* Cf. The Annals of Mathematics, Art. 10, p. 185.

When we make the substitution above, the single points on the curve are subjected to a certain *sliding.*

If we neglect the 2nd powers of ε, then each point, owing to the substitution, moves along a straight line ; and this is quite sufficient in the investigations concerning the calculation of the first variation.

11. Making this substitution in 1) we have :

$$I + \Delta I = \int_{t_0}^{t_1} F(x+\varepsilon\xi,\, y+\varepsilon\eta,\, x'+\varepsilon\xi',\, y'+\varepsilon\eta')\, dt\,,$$

which developed by Maclaurin's theorem is

$$= \int_{t_0}^{t_1} \left\{ F + \varepsilon\left[\frac{\partial F}{\partial x}\xi + \frac{\partial F}{\partial y}\cdot\eta + \frac{\partial F}{\partial x'}\xi' + \frac{\partial F}{\partial y'}\eta'\right] + \varepsilon^2(\ldots)\right\} dt\,.$$

Further (p. 189)

$$\Delta I = \varepsilon\delta I + \frac{\varepsilon^2}{1\,.\,2}\delta^2 I + \ldots\,;$$

hence equating coefficients of ε,

2)
$$\delta I = \int_{t_0}^{t_1} \left[\frac{\partial F}{\partial x}\xi + \frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial x'}\xi' + \frac{\partial F}{\partial y'}\eta'\right] dt\,.$$

But

$$\int_{t_0}^{t_1} \frac{\partial F}{\partial x'}\cdot\frac{d\xi}{dt}\cdot dt = \left[\frac{\partial F}{\partial x'}\cdot\xi\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \xi\,\frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right) dt\,;$$

so that 2) becomes

2′)
$$\delta I = \int_{t_0}^{t_1} \left\{ \left[\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right)\right]\xi + \left[\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial y'}\right)\right]\eta \right\} dt + \left[\frac{\partial F}{\partial x'}\cdot\xi + \frac{\partial F}{\partial y'}\eta\right]_{t_0}^{t_1}\,;$$

or

3)
$$\delta I = \int_{t_0}^{t_1} \{G_1\xi + G_2\eta\}\, dt + \left[\frac{\partial F}{\partial x'}\xi + \frac{\partial F}{\partial y'}\eta\right]_{t_0}^{t_1}\,,$$

where

$$G_1 = \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right),$$

$$G_2 = \frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial y'}\right).$$

12. *On the proof that* $G_1 = 0 = G_2$ *in order that the integral in question may have a maximum or a minimum value.*

ξ and η are functions of t which we may choose at pleasure. Among all the possible variations we choose those by which the end points P_0 and P_1 remain fixed; and hence at these points $\xi = 0 = \eta$.

We still further assume that the sliding takes place parallel to the axis x, i. e., that $\eta = 0$.

Then choosing

$$\xi = \left[\frac{\partial F}{\partial x} - \frac{d}{dt}\left[\frac{\delta F}{\delta x'}\right]\right](t - t_0)(t_1 - t),$$

it is seen that

$$\delta I = \int_{t_0}^{t_1} G_1^2 (t - t_0)(t_1 - t),$$

where G_1 is by supposition continuous.

In this expression for δI, all the terms are positive, so that δI can only be zero* when

$$G_1 = 0$$

along the whole curve.

In a similar manner, when we write

$$\eta = G_2 (t - t_0)(t_1 - t),$$

it is seen that

$$G_2 = 0$$

along the whole curve.

ξ and η must be continuous and also the differential quotients $\frac{d\xi}{dt}, \frac{d\eta}{dt}$ must exist in these investigations; and from our previous methods it is seen that the derivatives of x and y with respect to t up to the third order must exist, because the radius of curvature must constantly change and in a continuous manner.

In the following researches only derivatives up to the third order occur, so that the derivatives of x and y with respect to t need only exist up to this order.

13. *On the reduction of* G_1 and G_2 *to a single equation.*

We wrote (Art. 11)

$$G_1 = \frac{\partial F}{\partial x} - \frac{d}{dt}\left[\frac{\partial F}{\partial x'}\right];$$

and it was shown in Art. 8 that

$$F(x, y, x', y') = x'\frac{\partial F}{\partial x'} + y'\frac{\partial F}{\partial y'}.$$

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* Art. 2, p. 82.

Therefore

$$\frac{\partial F}{\partial x} = x' \frac{\partial^2 F}{\partial x' \partial x} + y' \frac{\partial^2 F}{\partial y' \partial x},$$

and

$$-\frac{d}{dt}\left[\frac{\partial F}{\partial x'}\right] = -\frac{\partial^2 F}{\partial x \partial x'} x' - \frac{\partial^2 F}{\partial y \partial x'} y' - \frac{\partial^2 F}{\partial x'^2}\frac{dx'}{dt} - \frac{\partial^2 F}{\partial y' \partial x'}\frac{dy'}{dt}.$$

Hence,

$$G_1 = y'\left[\frac{\partial^2 F}{\partial x \partial y'} - \frac{\partial^2 F}{\partial y \partial x'}\right] - \left[\frac{\partial^2 F}{\partial x'^2}\frac{dx'}{dt} + \frac{\partial^2 F}{\partial y' \partial x'}\frac{dy'}{dt}\right].$$

Writing $\frac{\partial^2 F}{\partial x'^2} = y'^2 F_1$ (see Art. 9), and $\frac{\partial^2 F}{\partial y' \partial x'} = -x'y'F_1$; and defining G by the equation

$$G = \frac{\partial^2 F}{\partial x \partial y'} - \frac{\partial^2 F}{\partial y \partial x'} - F_1\left[y'\frac{dx'}{dt} - x'\frac{dy'}{dt}\right],$$

it is seen that

$$G_1 = y' . G .$$

In a similar manner it may be shown that

$$G_2 = -x' . G .$$

14. One problem was so to determine x and y as functions of t that the integral

$$\int_{t_0}^{t_1} F(x, y, x', y')\, dt,$$

(where F is a given function of the four arguments) should be a maximum or a minimum.

In general $G = 0$ is the differential equation of the problem; however, $G_1 = 0$, $G_2 = 0$, $G = 0$ are three different forms of one and the same differential equation.

From formula 3) Art. 11 we have, when we write $G_1 = y' . G$ and $G_2 = -x' . G$,

$$\delta I = \int_{t_0}^{t_1} G\,(y'\xi - x'\eta)\, dt + \left[\xi \frac{\partial F}{\partial x'} + \eta \frac{\partial F}{\partial y'}\right]_{t_0}^{t_1};$$

where along the whole curve it is necessary that $G = 0$, if a maximum or a minimum value of the given integral is to enter.

We thus have a differential equation of the second order for the determination of the required curve.

15. In the differential equation

$$G = 0$$

the second derivatives of the coordinates with respect to t appear only in the simple determinative form:

$$y'x'' - x'y''.$$

The radius of curvature at any point is:

$$\rho = \frac{\left[1 + \left[\frac{dy}{dx}\right]^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\frac{1}{(dx)^3}\,ds^3}{\frac{d}{dx}\left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right]} = \frac{\frac{ds^3}{dx^3}}{\frac{y''x' - y'x''}{x'^3}} = \frac{\left[\frac{ds}{dt}\right]^3}{y''x' - y'x''};$$

or if k denotes the curvature:

$$y'x'' - x'y'' = -k\left[\frac{ds}{dt}\right]^3.$$

Writing this in $G = 0$, it is seen that through the equation $G = 0$ a definite relation is expressed between the curvature of the curve at a definite point, the coordinates, and the direction of the tangent of the curve at this point.

The curve is definitely determined excepting two constants through the differential equation $G = 0$, and the two constants are determined through the two limiting conditions.

16. The equations $G_1 = 0 = G_2$ are often more convenient than $G = 0$; especially is this the case if F does not contain explicitly one of the two quantities x and y, since then, if x, say, is wanting, then $\frac{\partial F}{\partial x} = 0$, and from $G_1 = 0$

$$\frac{d}{dt}\left[\frac{\partial F}{\partial x'}\right] = 0,$$

an equation which is at once integrable.

17. We will now free ourselves from some of the previous limiting assumptions. First of all, with regard to the choice of the special variations of the curve, it is clear that owing to this choice the generality of the derived condition, which is necessary for the existence of a maximum or of a minimum, does not suffer any loss; since ΔI is for all variations to be continuous and > 0 (or < 0) and under these variations our special variations are contained, so that the condition that has been found is certainly necessary, however the variations may have been chosen.

We have further assumed that x', y', i. e., the direction of the curve, experiences no sudden changes ; and therefore the curve is to consist of one regular trace. This assumption we now drop, and only retain the assumption that the curve is composed of regular portions, so that consequently the direction of the tangent need not change in a continuous manner at all places. Then each portion of curve must satisfy the differential equation.

18. Since for example, if the curve AB consists of two such portions AC and CB, then among all the possible variations of AC, there are also these, by which CB remains unaltered and only AC is subject to variation.

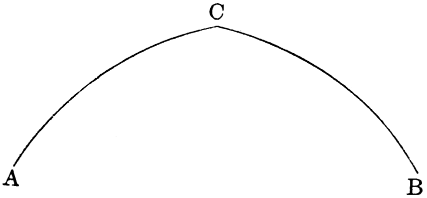

FIG. 2.

Now the variation of the integral depends only upon the variation of AC, and we conclude as above, that this portion of the curve must satisfy the differential equation $G = 0$. The same is true of CB ; therefore, along the whole curve the equation $G = 0$ must be fulfilled.

In the case of such a transition from one regular portion to another there remains yet for the integration of $G = 0$ a very important theorem, the necessity of which we will make clear by means of an example, especially because in the text-books it is overlooked.

Grant that F (as in the first problem, p. 183) is independent of x, and for this reason we will make use of $G_1 = 0$ instead of $G = 0$.

From the equation :

A) $$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial x'} = 0\,,$$

we have here at once

B) $$\frac{\partial F}{\partial x'} = \text{const (with respect to } t)\,.$$

However, it is not at all excluded, that $\frac{\partial F}{\partial x'}$ makes anywhere a sudden spring, since we do not know whether this function is continuous, and whether we can, therefore, go from the equation A) to B) in the sense that the constant has everywhere the same value.

One would most naturally expect, since x', y' make springs between t_0 and

t_1, that also $\frac{\partial F}{\partial x'}, \frac{\partial F}{\partial y'}$ will be discontinuous, and therefore the constant in different intervals would have a different value.

Hence the more important is the following theorem: *Even if x', y', and therefore the direction of the curve, suffers an alteration which is due to a sudden spring, nevertheless $\frac{\partial F}{\partial x'}, \frac{\partial F}{\partial y'}$, change in a continuous manner along the whole curve.*

A discontinuity could only take place at such a point of transition where x', y' suffer a spring.

19. If to the value t_1 of t there corresponds such a spring, then on both

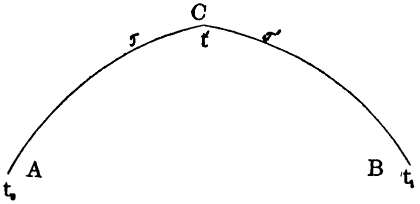

FIG. 3.

sides of t' we take 2 points τ and τ' in such a manner that within the portions $\tau \ldots t'$ and $t' \ldots \tau'$, there is no other discontinuity in the direction of the curve. Then a possible variation of the curve is also the one by which $t_0 \ldots \tau$ and $\tau' \ldots t_1$ remain unaltered, and only the portion $\tau \ldots \tau'$ is varied. Here τ and τ' remain fixed while t_1 is subjected to any kind of sliding.

The variation of the integral,

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt$$

then depends only upon the variation of the integrals

$$1) \qquad \int_{\tau}^{t'} F(x, y, x', y')\, dt + \int_{t'}^{\tau'} F(x, y, x', y')\, dt\,.$$

And in a manner similar to that employed above, the 1st variation of 1) must vanish, so that we have the equation:

$$0 = \int_{\tau}^{t'} G\,(y'\xi - x'\eta)\, dt + \int_{t'}^{\tau'} G\,(y'\xi - x'\eta)\, dt$$
$$+ \left[\frac{\partial F}{\partial x'}\,\xi + \frac{\partial F}{\partial y'}\,\eta\right]_{\tau}^{t'} + \left[\frac{\partial F}{\partial x'}\,\xi + \frac{\partial F}{\partial y'}\,\eta\right]_{t'}^{\tau'}.$$

But since G must vanish along the whole curve, as we have already seen, this equation reduces to

2) $$\left[\frac{\partial F}{\partial x'}\xi + \frac{\partial F}{\partial y'}\eta\right]_{\tau}^{t'} + \left[\frac{\partial F}{\partial x'}\xi + \frac{\partial F}{\partial y'}\eta\right]_{t'}^{\tau'} = 0\,.$$

For the values τ and τ', they being now supposed belonging to fixed points, the expressions on the left vanish, but we do not know but that each of the two quantities $\frac{\partial F}{\partial y'}$, $\frac{\partial F}{\partial x'}$ have different values at t' according as we approach the point t' from τ or from τ'.

We suppose that this is the case, and consequently we make a distinction between

$$\left[\frac{\partial F}{\partial x'}\right]_{t'}^{-} \text{ and } \left[\frac{\partial F}{\partial x'}\right]_{t'}^{+}, \text{ also between } \left[\frac{\partial F}{\partial y'}\right]_{t'}^{-} \text{ and } \left[\frac{\partial F}{\partial y'}\right]_{t'}^{+}.$$

Then equation 2) becomes

$$\left\{\left[\frac{\partial F}{\partial x'}\right]_{t'}^{-} - \left[\frac{\partial F}{\partial x'}\right]_{t'}^{+}\right\}(\xi)' + \left\{\left[\frac{\partial F}{\partial y'}\right]_{t'}^{-} - \left[\frac{\partial F}{\partial y'}\right]_{t'}^{+}\right\}(\eta)' = 0$$

where $(\xi)'$ and $(\eta)'$ are the values of ξ and η at the point t'. But since these values are quite arbitrary it follows that their coefficients in the above expression must respectively vanish, so that:

3) $$\left[\frac{\partial F}{\partial x'}\right]_{t'}^{-} = \left[\frac{\partial F}{\partial x'}\right]_{t'}^{+} \text{ and } \left[\frac{\partial F}{\partial y'}\right]_{t'}^{-} = \left[\frac{\partial F}{\partial y'}\right]_{t'}^{+};$$

that is, *the quantities* $\frac{\partial F}{\partial x'}$ and $\frac{\partial F}{\partial y'}$ *vary by the transition from the one regular part of the curve to the other in a continuous manner, even if* x' *and* y' *at this point make springs.*

Remark. For the special case that was given above as an example, in which F is independent of x, we conclude at once, that the constant of the differential equation, $\frac{\partial F}{\partial x} =$ const., has everywhere the same value.

This theorem may be often used to prove that the direction of the curve in question nowhere changes in a discontinuous manner, that the curve does not consist of several regular portions, but of one single regular trace.